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Borel convergence of the variationally improved mass expansion and dynamical symmetry breaking

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Abstract. A modification of perturbation theory, known as the delta expansion (variationally improved perturbation), gave rigorously convergent series in some $D = 1$ models (oscillator energy levels) with factorially divergent ordinary perturbative expansions. In a generalization of the variationally improved perturbation technique appropriate to renormalizable asymptotically free theories, we show that the large expansion orders of certain physical quantities are similarly improved, and prove the Borel convergence of the corresponding series for $m_{\rm v} \lesssim 0$, with $m_{\rm v}$ the new (arbitrary) mass perturbation parameter. We argue that non-ambiguous estimates of quantities relevant to dynamical (chiral) symmetry breaking in QCD are possible in this resummation framework.

1 Introduction

A "first principle" determination of the order parameters characterizing dynamical (e.g. chiral) symmetry breaking (χSB) in asymptotically free theories (AFT) like QCD is traditionally considered inaccessible (except perhaps from lattice calculations), due to three main obstacles:

(i) order parameters, like the quark condensate $\langle\bar{q}q\rangle^{1/3}$, are expected to be of $\mathcal{O}(\Lambda_{\rm QCD})$, so that the coupling at such scale is large: ordinary perturbative expansion is invalidated.
(ii) At arbitrary perturbative order, $\langle\bar{q}q\rangle$ and other χSB order parameters vanish anyhow in the massless limit: their chiral limits are (perturbatively) trivial.
(iii) A more subtle but equally important argument is that attempts to extract genuine non-perturbative contributions to such quantities meet with inherent ambiguities, as indicated by the (infrared) renormalon singularities of perturbative expansions [1,2]. Conventional wisdom thus treats $\langle\bar{q}q\rangle$ and other non-perturbative condensates as *parameters* of a systematic operator product expansion (OPE)[1], as best illustrated in the SVZ formalism [3].

Yet in many field theory models, definite non-perturbative results may be obtained from an appropriately resummed (but different) expansion, like the $1/N$ expansion [4,5]. There also exist powerful summation techniques, like the Borel method [6,2] which, even for non-Borel summable expansions like typically in QCD, gives nevertheless precious information on the nature of (power-like) non-perturbative contributions to a given physical quantity. An alternative summation method, known as the delta expansion (DE) or "variationally improved perturbation" (VIP) [7,8], is based on a reorganization of the interaction Lagrangian making it depend on arbitrary adjustable parameters, to be fixed by some optimization prescription. In various models DE-VIP exhibits (though often rather empirically) an improved convergence of the perturbative expansion. Moreover in some $D = 1$ models, for which the anharmonic oscillator is typical, DE-VIP is equivalent [9] to the "order dependent mapping" (ODM) method [10], and optimization is equivalent to a rescaling of the adjustable oscillator frequency (mass) with perturbative order, which can essentially suppress the factorial asymptotic behavior of ordinary perturbative coefficients. Such a procedure was proven rigorously to converge [9, 11] (for an adequately rescaled mass) toward the exact result, e.g. for the oscillator energy levels [12] and related quantities.

Here we reconsider a variant of DE-VIP adapted to higher dimensional renormalizable theories, proposed some time ago [13–15]. The basic idea is to perform a modification of the perturbative expansions in two stages: first exploiting specific renormalization group (RG) properties, which transform the ordinary expansion (in a coupling g) of certain physical quantities, depending only on g and on a mass m, in the alternative form of "mass power" expansions (MPE) in $(\hat{m}/\Lambda)^\alpha$ [$\hat{m}$ is the renormalization scale-invariant mass, Λ the basic RG scale and α is given by known RG coefficients]. This construction resums the RG dependence to all orders (at least in specific schemes), and most interestingly exhibits a non-trivial massless (chi-

[1] Unlike the gluon condensate, the presence of χSB condensates like $\langle\bar{q}q\rangle$ in OPE's is however not directly inferred by infrared renormalons, these being screened by chiral symmetry [1], cf. argument (ii) above. We will see that renormalons and argument (iii) are nevertheless relevant to χSB quantities in our context

ral) limit [14,15] for DSB (χSB) order parameters, or for analogous quantities like the "mass gap" in $D = 2$ models [13]. However, such a result turns out to be well defined only in the approximation of neglecting all the purely perturbative (non-RG) dependence. When arbitrary large orders of the complete (non-log) perturbative series are included, our naive mass gap result is plagued with ambiguities originating mainly from renormalon singularities, cf. point (iii) above, as we shall examine in more detail here.

However, in a second stage, an appropriate version of the (order dependently rescaled) DE-VIP can be performed on the complete MPE series in $\hat{m}/\Lambda$, essentially replacing the true physical mass by an arbitrary adjustable mass parameter. In this note we mainly investigate the large order behavior of the resulting "variational" expansion in this mass parameter[2]. We find that it produces a renormalization scheme (RS) dependent factorial *damping* of the original perturbative coefficients at large orders. Yet, unlike the oscillator case, the damping is generally not sufficient to make the DE-VIP series readily convergent, in the standard perturbative regime, when the generically expected renormalon singularities are taken into account. But we show that the series can be *Borel* convergent, if approaching the chiral limit with the arbitrary mass parameter $\mathrm{Re}[\hat{m}]\lesssim 0$. These results apply formally a priori to any (asymptotically free) renormalizable models. Some concrete examples are the $D = 2$ $O(N)$ Gross–Neveu (GN) model [5] (where the mass gap is known exactly [17]); or a $D = 4$ gauged AFT with n_f massless fermions like QCD, where the expected [18] $SU(n_f)_{\mathrm{L}} \times SU(n_f)_{\mathrm{R}} \to SU(n_f)_V$ breaking is exhibited via non-perturbative order parameters.

2 Transmuted mass expansion and mass gap

In this and the next section we essentially summarize some of the constructions in [13–15]. To simply illustrate the first stage, consider in a "generic" AFT the first RG order evolution for the renormalized "current" mass:

$$M_1 = m(\mu)[1 + 2b_0 g^2(\mu)\ln(M_1/\mu)]^{-\gamma_0/(2\,b_0)}, \qquad (1)$$

where b_0, γ_0 are one-loop RG coefficients, with $b_0 > 0$ for an AFT [our normalization is $\beta(g) = -b_0 g^3 - b_1 g^5 - \cdots$, $\gamma_m(g) = \gamma_0 g^2 + \gamma_1 g^4 + \cdots$], and the *self-consistent* condition $M_1 \equiv m(M_1)$ defines M_1. Now, (1) equivalently reads

$$M_1(\hat{m}) = \hat{m}[\ln(M_1/\Lambda)]^{-A} \equiv \hat{m}F^{-A}, \qquad (2)$$

with $\Lambda = \mu\exp[-1/2b_0 g^2(\mu)]$ the RG invariant scale, $\hat{m} \equiv m(\mu)[2b_0 g^2(\mu)]^{-A}$ the scale-invariant mass ($A \equiv \gamma_0/(2b_0)$), and in (2)

$$F(\hat{m}/\Lambda) \equiv \ln(\hat{m}/\Lambda) - A\ln F = AW[A^{-1}(\hat{m}/\Lambda)^{1/A}], \qquad (3)$$

where the Lambert [19] function, $W[x] \equiv \ln x - \ln W$, is plotted in Fig. 1. Equation (3) has the remarkable property

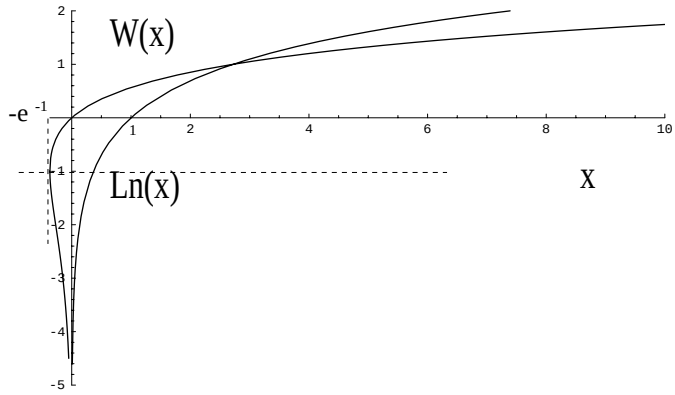

Fig. 1. The Lambert function $W(x)$ compared to $\ln x$

$F \simeq (\hat{m}/\Lambda)^{1/A}$ for $\hat{m} \to 0$, in contrast with the ordinary Log (see Fig. 1), which is asymptotic to $F(\hat{m}/\Lambda)$ for $\hat{m} \gg \Lambda$. More precisely, on its principal branch (which is real-valued for real arguments), F has an alternative series expansion:

$$F(x) = \sum_{p=0}^{\infty}\left(\frac{-1}{A}\right)^p \frac{(p+1)^p}{(p+1)!}x^{(p+1)/A} \qquad (4)$$

with finite convergence radius $R_{\mathrm{c}} = \mathrm{e}^{-A}A^A$. $M_1(\hat{m})$ in (2) thus exhibits different branches according to the values of the RG parameter A (see Fig. 2). Now, for most values of A, there is only one branch which for *real* $\hat{m}$ values, is real and continuously matching the asymptotic perturbative behavior of F at large $\hat{m}$: the one giving a non-zero "mass gap" $M_1 = \Lambda$ for $\hat{m} \to 0$ (region I in Fig. 2). Algebraically, the mass is obtained by expanding (4) in (2):

$$M_1(\hat{m} \to 0) = \hat{m}[(\hat{m}/\Lambda)^{1/A} + \cdots]^{-A} = \Lambda(1 + \mathcal{O}(\hat{m}/\Lambda)^{1/A}), \qquad (5)$$

which may be viewed as a generalization (for $m \neq 0$) of the "dimensional transmutation". Note that (5) readily reproduces, e.g., the GN $O(N)$ model mass gap in the large N, $m \to 0$ limit (where $A \to 1$ for $N \to \infty$), traditionally obtained in a different way [5]. Equation (5) is, however, not a proof of dynamical χSB: rather, *if* χSB occurs, the fact that any mass is proportional to Λ is consistently incorporated by the properties of $F(\hat{m})$ for any $\hat{m}$, which provides an explicit bridge between the "non-perturbative" $\hat{m}\lesssim\Lambda$ regime, where F has a power expansion (4), and the usual short distance perturbative $\hat{m} \gg \Lambda$ (Log) regime. A crucial point is the difference between the usual effective coupling $g^2(p^2) \equiv 1/[b_0\ln(p^2/\Lambda^2)]$, having a Landau pole at $p^2 = \Lambda^2$, and $F^{-1}(\hat{m})$ here, having its pole at $\hat{m} = 0$, governing the massless limit (5) of the (pure RG) mass gap (2)[3]. Accordingly along the continuous branch I, $M_1(\hat{m})$ has no singularity for $0 < \hat{m} < \infty$, as is clear from (5) and Fig. 2.

[2] See also [16] for a preliminary discussion

[3] $W(x)$ appears in various branches of physics [19], in particular recently also in the QCD and RG context [20]. Yet its connection with the non-trivial chiral limit (5) was unnoticed before [13–15], to the best of our knowledge

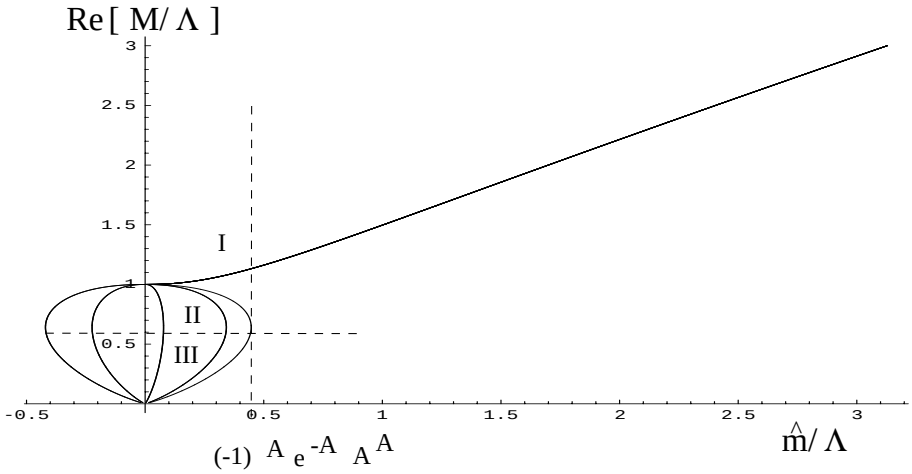

Fig. 2. The different branches of M_1 in (2), for $A = 4/9$ (corresponding to first RG order QCD with three active quark flavors)

3 Pole mass gap and other DSB quantities

Equation (2) also defines a (lowest order) "pole" mass, being scale invariant to all orders (and gauge invariant as well, if gauge symmetry is relevant, as in QCD), thanks to its continued fraction form in M_1. Yet the genuine pole mass is not given simply by (1), as it includes non-log perturbative and RG contributions of arbitrary higher orders, though for most theories one only knows at present the perturbative series up to the second or third order coefficients, like e.g. in the GN model [22,13] or QCD [21].

A generalization of (2), perturbatively consistent [15] with the usual pole mass, can be defined[4]:

$$M^P(\hat{m}) = 2^{-C}\hat{m}F^{-A}[C+F]^{-B}\sum_{n=0}^{\infty} d_n(2b_0F)^{-n}, \quad (6)$$

with

$$F = \ln[\hat{m}/\Lambda] - A\ln F - (B-C)\ln[C+F] \quad (7)$$

and

$$A = \frac{\gamma_1}{2b_1}, \quad B = \frac{\gamma_0}{2b_0} - A, \quad C = \frac{b_1}{2b_0^2}. \quad (8)$$

$F(\hat{m})$ in (6) and (7) resums the RG dependence in $\ln[\hat{m}]$ at two-loop order exactly (or even to all orders in the scheme $b_i = 0, \gamma_i = 0\ \forall i \geq 2$). Most interestingly, similarly to (4) F also has an (A, B, C dependent) expansion in $(\hat{m}/\Lambda)^{1/A}$ for sufficiently small $\hat{m}$, with A now defined in (8). The coefficients d_n implicitly include the non-log perturbative contributions from n-loop graphs (generically dominant, as discussed below), plus eventually (subdominant) contributions from higher RG orders.

A similar construction can be performed for other physical quantities, at least those depending only on m and g. Examples are the perturbative expansion of the GN model vacuum energy [13], or in QCD the χSB order parameters F_π/Λ (the pion decay constant) and $\langle\bar{q}q\rangle(\mu)/\Lambda^3$ [14,15].

[4] Strictly, (6) applies only if $C \equiv b_1/(2b_0^2) \geq 0$. If $C < 0$ (as in the $O(N)$ GN model, corresponding to an infrared fixed point at $g^2 = -b_0/b_1 > 0$), an alternative appropriate RG summation can be defined [13,25]

Now in (6), there are crucial differences with the "pure RG" mass gap, (2).

(1) The pole mass (or other physical quantities similarly) is infrared finite, gauge- [23], scale- and scheme-invariant, but the relation between the pole mass and e.g. the running mass is scheme dependent, which is manifest here by the RS dependence in (6) of the following: the perturbative coefficients d_n; the RG coefficients A, B in (8); and of course Λ.

(2) The dominant contributions d_n in (6) behave rather generically as [24,2]

$$d_{n+1} \underset{n\to\infty}{\sim} (2b_0)^n n!, \quad (9)$$

so that the series (6) is badly divergent for any $\hat{m} > 0$, and not even Borel summable: such a factorial growth of the perturbative coefficients, with no sign alternation, implies [2] ambiguities of $\mathcal{O}(\Lambda)$; this we reexamine within the present context in Sect. 5. The $O(N)$ GN model mass gap, at order $1/N$, also exhibits infrared renormalons similar [27] to (9), if considering only its naive perturbative expansion. In QCD, insertions of the (resummed) gluon propagator in the F_π or $\langle\bar{q}q\rangle$ perturbative expressions potentially give factorially growing asymptotic coefficients: while usually considered irrelevant in the $m \to 0$ limit (cf. argument (ii) above), the factorial behavior survives a priori in our construction due to the non-trivial chiral limit[5].

4 Variationally improved mass expansion

We shall examine now how to possibly cure the latter potential ambiguities of such a resummation of DSB quantities, by combining the previous MPE series construction leading to e.g. (6) with a specific form of delta expansion. As mentioned in the introduction, DE-VIP is essentially a reorganization of the interaction terms of the Lagrangian.

[5] The form of those "χSB parameter renormalons" will be discussed elsewhere [25,26]

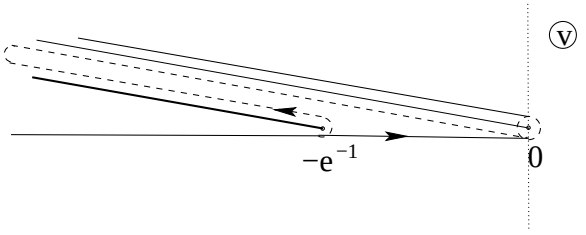

Fig. 3. Singularities and equivalent integration contours in the v plane, for $A = 1$

More specifically, here we define a (linear) DE as the substitution

$$m(\mu) \to (1-\delta)m_{\rm v}, \quad g^2(\mu) \to \delta g^2(\mu) \tag{10}$$

within perturbative expressions at arbitrary order, where $m(\mu)$ is the renormalized Lagrangian mass (in e.g. the $\overline{MS}$ scheme), δ is the new expansion parameter, and $m_{\rm v}$ an *arbitrary* adjustable mass. Equation (10) is equivalent to adding to and subtracting from the massless Lagrangian a "trial" mass term $m_{\rm v}$ [δ interpolating between the free ($\delta = 0$) and the interacting *massless* Lagrangian ($\delta = 1$)], and is entirely compatible with renormalization [13] and gauge-invariance [15]. The procedure then usually [8] is to take the limit $\delta \to 1$ *after* performing a perturbative expansion of the relevant physical quantities to fixed order δ^k, exhibiting a residual $m_{\rm v}$ dependence, so that an optimization prescription, typically the "principle of minimal sensitivity" (PMS) [7], can be applied with respect to $m_{\rm v}$. However, we go here a step beyond this standard PMS usage by following more closely the logic that leads to rigorous convergence properties of the DE method for the oscillator.

In what follows we only investigate for simplicity the mass gap (6), but our construction can easily be generalized to similar DSB quantities. After applying substitution (10), $M^P(\hat{m},\delta) \equiv \sum_k a_k(\hat{m})\delta^k$ can be most conveniently directly resummed, for $\delta \to 1$, by contour integration [13] around $\delta = 0$, to arbitrary order K: an appropriate change of variable allowing one to study the $m(\mu) \to 0$ (equivalently $\delta \to 1$) limit in (10) is

$$\delta \equiv 1 - v/K, \quad m_{\rm v} = K^\gamma \hat{m}_{\rm v}. \tag{11}$$

Equation (11) is simply a convenient way of parameterizing how rapidly the Lagrangian mass $m(\mu) \to 0$ limit is reached (as controlled by $\gamma \le 1$) as a function of the (maximal) delta expansion order K. Similarly to [9] the point is to adjust the rates at which $m(\mu) \to 0$ ($\delta \to 1$) and $K \to \infty$ are simultaneously reached, with no a priori need of invoking an explicit optimization principle.

The final contour integral summation takes a simple form, for $K \to \infty$:

$$M^P/\Lambda \sim \sum_{n=0}^{N} \frac{1}{2\pi {\rm i}} \oint {\rm d}v {\rm e}^{v/m''} F^{-A}[v] d_n (2b_0 F[v])^{-n}, \tag{12}$$

where $m'' \equiv \hat{m}_{\rm v}/\Lambda$, N is the maximal perturbative order, and after deformation the contour encircles the semi-axis ${\rm Re}[v] < 0$ (see Fig. 3), and also for simplicity we fix from now on the scaling parameter in (11) to its maximal value ($\gamma = 1$) still compatible with massless limit (for $\hat{m}_{\rm v} \to 0$). [The general γ scaling (11) can be analyzed [25,26] in a way more similar to the oscillator [9,11], i.e. without the peculiar contour δ summation (12), but it largely complicates the algebraic analysis for renormalizable theories. In (12) we also omit some overall constant factors (due e.g. to the Λ definition) irrelevant for convergence properties, and temporarily made a RS choice such that $B \equiv 0$ in (6)–(8), rendering certain algebraic expressions below more tractable, without much loss of generality.] Equation (12) can be well approximated analytically (at least for slightly restricted RS choices, as indicated above and further below):

$$\begin{aligned} &M^P/\Lambda \\ &\sim 1 + \frac{1}{2b_0} \sum_{q=1}^{N} \left[\sum_{p=0}^{N-q} \frac{\Gamma[p+q](p+q+A)(q+A)^{p-1}}{A^p \Gamma[1+p]\Gamma[1+q/A]} \right] \\ &\times (m'')^{-q/A}, \end{aligned} \tag{13}$$

where we assumed the leading renormalon behavior (9)[6], and we used essentially (4) together with $\oint {\rm d}v {\rm e}^v v^r = 2\pi {\rm i}/\Gamma[-r]$, $\forall r$.

Now, some restrictions apply to (13): first, the sum over p is bounded iff.

$$1/A \in \mathbf{N}^*, \tag{14}$$

which we assume for simplicity from now on. This is not very restrictive, except that for arbitrary AFT it is generally not possible that A both satisfies (14) *and* $B = 0$ in (8), as assumed in (12). But the more general scheme $B \neq 0$ simply makes (13) algebraically more involved, without affecting the asymptotic behavior and convergence properties discussed below.

Second, strictly (13) is valid only asymptotically, for sufficiently large N: due to the finite convergence radius of the expansion (4), interchanging the sum in (4) and integration in (12) is not rigorously justified. However, when (14) holds, the former branch point $v = 0$ is simply a pole, which allows one to choose an equivalent contour of arbitrarily small radius around $v = 0$, thus always inside the convergence radius of (4) (see the dashed small circle contour in Fig. 3). So, only the simple pole terms v^{-1} contribute to (12), which finally sum up to (13). The extra contribution (around the cut at $v = -{\rm e}^{-1}$, e.g. for $A = 1$) gives the difference between the "exact" integral (12) and the expansion (13) and can be evaluated numerically. These contributions are easily shown for $A = 1$ to contribute as $\mathcal{O}({\rm e}^{-(em'')^{-1}})h[N]$ relative to (13), where $h[N]$ rapidly decreases for $N \to \infty$. Thus for large enough

[6] The original $n!$ coefficients in (9) correspond to $\Gamma[p+q]$ in (13). Higher order refinements on the infrared renormalon structure may easily be implemented: they essentially replace $(n-1)! \to \Gamma[n+b_1/(2b_0^2)](1+r_{\rm RS}/n+...)$ where $r_{\rm RS}$ depends on RS via b_2 etc. [2], without affecting the convergence properties discussed below

N (and/or small m'') those contributions are unessential for the convergence properties discussed below.

The announced factorial damping of coefficients, as compared to the original perturbative expansion, is explicit in (13). Yet, closer examination indicates that the damping is insufficient to make this series for $N \to \infty$ readily convergent. Before considering the asymptotic behavior of the full series (13), it is instructive to examine the $p = 0$ terms, behaving as

$$\sim \sum_q^N \Gamma[q]/\Gamma[1+q/A](m'')^{-q/A}. \tag{15}$$

The denominators in (15) overcompensate the numerator factorials iff.

$$0 < A \leq 1, \tag{16}$$

where for $D > 1$ AFT, A is RS dependent, as discussed in Sect. 3. Thus, if our series would only consist of terms of the form (15), the solution would be simply to perform appropriate scheme changes $A \to A'$ in (6) and (13) etc., so that a damping of coefficients larger than (or equal to) the factorial growth would make the series convergent. [For such RS changes in A one should consistently derive the corresponding change in e.g. the first few perturbative coefficients d_1, etc., and in Λ, but this one-parameter RS change does not reintroduce any factorial behavior in d_n at large orders. Moreover, if (16) holds, any generic infrared (or ultraviolet) renormalon behavior of the form $\sim r^n n!$ with r arbitrary [2] is damped similarly.]

Unfortunately, the large N behavior of (13) differs from the simple "oscillator form" (15), due to the $p \geq 1$ terms in the expansion (4) reminiscent of RG properties. For any low $p \ll N$, renormalon factorials are still overcompensated if $A \leq 1$, but the $\Gamma[1+q/A]$ damping decreases in strength as p increases, giving increasing contributions to the sum over p. All in all, the leading contributions to the coefficients of (13) occur at intermediate values of p. Nevertheless, the idea of damping factorials from an appropriate RS choice does survive, when the series (13) is Borel transformed, as examined in the next section.

5 Borel convergence of DE-VIP

A Borel integral slightly adapted to our case reads

$$\mathrm{BI}(\hat{m}) \equiv \tilde{M}^P(\hat{m}) = 2^{-C}\hat{m}F^{1-A} \times (C+F)^{-B}\int_0^\infty \mathrm{d}t\mathrm{e}^{-Ft}\left[1+(2b_0F)^{-1}\sum_{n=0}^\infty t^n\right], \tag{17}$$

which would be (asymptotically) equal to (6) by a formal expansion[7] (upon assuming (9)), would the pole at $t_0 = 1$ not make the integral (17) ill defined. One should make a choice in deforming the contour e.g. above (or below) the pole, which results in an ambiguity, easily calculated to be $\mathcal{O}(\mathrm{e}^{-F})$. Since $F \sim \ln[\hat{m}/\Lambda]$ for $\hat{m} \gg \Lambda$, an $\mathcal{O}(\Lambda/m)$ ambiguity [2] for the "short distance" $(M, \hat{m} \gg \Lambda)$ pole mass is recovered. But in our construction (4) allows one to trace the behavior of F all the way down to $\hat{m} \to 0$, where $F \to 0$: there the ambiguity becomes $\mathcal{O}(1)$, and the naive RG-summed mass gap (5), which is $\mathcal{O}(\Lambda)$, gets an ambiguity of the same order, as announced.

Now for any given choice of contour avoiding the pole (or cut [2] at higher RG order) in the Borel plane t, let us apply the DE-VIP as defined in Sect. 4, introducing the δ expansion and contour resummation as in (12), this time on the Borel integral (17). Interchanging the contour and Borel integrals, one can find after some algebra the asymptotic behavior for $N \to \infty$:

$$\tilde{M}^P_{\mathrm{var}}(m'') \sim \Lambda\left[1+\int_0^\infty \frac{\mathrm{d}t}{2b_0}\sum_q^\infty \frac{(t^A\mathrm{e}^t/m'')^{q/A}}{\Gamma[1+q/A]}\right], \tag{18}$$

where we neglected for simplicity here the two-loop RG dependent $(C+F)^{-B}$ term in (17), as it does not affect the asymptotic behavior. It thus appears that the asymptotic behavior of the Borel integrand in (18) is that of an entire series (at least for $A > 0$), i.e. with no poles for $0 < t < \infty$. More precisely, the pole at $t_0 = 1$ in the original (standard) Borel integrand has been pushed to $t_0 \to +\infty$ due to the factorial damping, so that the Borel integral is no longer ambiguous. However, the integral (18) is not convergent, at least for $\mathrm{Re}[m''] > 0$, so that the series is not Borel summable for standard (perturbative) m'' values.

But conversely, the integral in (18) can converge, for $\mathrm{Re}[m''] < 0$. This is the case at least for $A = 1$, which can always be chosen by an appropriate and simple RS change, as previously explained. Now, since $m'' \equiv m_{\mathrm{v}}/\Lambda$ is an arbitrary parameter (and physical quantities anyway only depend on m^2 in relativistic field theories), it should be legitimate to reach the chiral limit $m'' \to 0$, of main interest here, within the Borel convergent half-plane $\mathrm{Re}[m''] < 0$. For $A \neq 1$, one may also choose the arbitrary parameter m'' with $\mathrm{Re}[(m'')^{1/A}] < 0$ such that (18) converges, though this is not always possible for any arbitrary A values. This is however only an artifact of our simplest choice of the δ expansion summation defining the DE-VIP series and leading to (18): for instance, an appropriate (A dependent) generalization of (10)–(11) defining the DE-VIP expansion, directly leads to a Borel convergent series independently of the A values [25]. Moreover, as already mentioned the function $F(\hat{m})$ in (3) is well defined (analytic) for any A values in a circle of radius $\mathrm{e}^{-A}A^A$ around zero (and for $A = 1$ the only singularity is at $F = -1$ i.e. $\hat{m}/\Lambda = -\mathrm{e}^{-1}$, cf. Fig. 1). The higher RG order F in (7) has similar properties, with finite (but RS dependent [15]) convergence radius around zero. Thus, one can choose $\mathrm{Re}[m''] \lesssim 0$ and/or equivalently $\mathrm{Re}[F(m'')] < 0$, while the mass gap $\mathrm{Re}[M(F)] \equiv \mathrm{Re}[M(\hat{m})]$ always remains positive, see Fig. 2.

[7] We define the Borel transform (integrand of (17)) by dividing the series coefficients by $(n-1)!$ for convenience. Also, the summed RG dependence $\hat{m}F^{-A}(C+F)^{-B}$, having no factorial behavior, is factored out of the Borel transform

Actually, one can see directly the Borel summability of (17), independently of A, without the need to consider the DE-VIP expansion: if $F < 0$, $F \equiv -|F|$ simply produces the adequate sign alternation in the factorially growing coefficients. More precisely, a straightforward calculation of (17) for $\mathrm{Re}[F] < 0$ (again neglecting the two-loop RG dependence C, irrelevant to asymptotic properties), gives

$$\tilde{M}^P/\Lambda \sim \mathrm{e}^{-|F|} + \frac{1}{2b_0}\mathrm{Ei}(-|F|), \tag{19}$$

where the exponential integral function $\mathrm{Ei}(-x)$ has a well-defined (sign alternated) asymptotic expansion for $x > 0$.

Note however that the DE-VIP expansion, leading to (18), appears to improve further the asymptotic behavior, at least for $A = 1$, as compared to (19), due to the extra factorial damping. For $A = 1$ and $\mathrm{Re}[m''] \lesssim 0$ (18) becomes after integration

$$\tilde{M}^P_{\mathrm{var}}(m'') \sim \mathrm{const.}\Lambda(1 + f(|m''|)), \tag{20}$$

where $f(|m''|) \to 0$ exponentially fast. The (here unspecified) overall constant in (20) originates essentially from RG dependence, involving non-trivial factors such as 2^{-C} etc. at second RG order, cf. (6) and (17).

We have thus obtained Borel convergence for a certain range of the arbitrary mass, strictly only for $\mathrm{Re}[m''] < 0$, but in which in addition the purely perturbative contributions are small and even vanishing in the chiral $|m''| \to 0$ limit. This does not mean, though, that our final DE-VIP result is completelly independent of the perturbative information, since the above series are only asymptotic to the exact series. Rather, it suggests that the "non-perturbative" result in the chiral limit may be essentially determined by pure RG properties, plus eventually the very first few perturbative terms, but not influenced by details of the large perturbative orders. Note indeed that only from the properties of F around $F \lesssim 0$ (thus independently of the DE-VIP construction) the Borel sum in (19) reproduces at least qualitatively the asymptotic behavior of the *exact* $1/N$ result in the $O(N)$ GN model[8]: in this model the $2b_0$ in (19) is more precisely replaced by $N - 2$, and the exact result [17] has an asymptotic expansion [27] similar to (19), except for a finite term γ_E, which not surprisingly cannot be guessed by our simple Borel summation of the (leading) renormalon behavior in (17).

6 Discussion

Though renormalon ambiguities are perturbative artifacts expected to disappear (or more precisely to cancel out with OPE contributions) in truly non-perturbative calculations [1,2,28,27], such explicit cancellations are generally inaccessible for theories like QCD. Rather, the peculiar damping mechanism of factorial divergences exhibited here is intuitively due to the fact that our reorganization of the perturbative expansions makes those much more similar to the oscillator energy level expansion, exhibiting a dependence on $\hat{m}_{\mathrm{v}}/\Lambda$, (4), which is *power-like* (rather than log-like) for sufficiently small $\hat{m}_{\mathrm{v}}$. Moreover, the adjustable parameter $\hat{m}_{\mathrm{v}}/\Lambda$ may be order dependently rescaled, or can take arbitrary values, in particular $\mathrm{Re}[\hat{m}_{\mathrm{v}}/\Lambda] < 0$ producing sign alternation of factorial coefficients. The DE-VIP expansion appears in that way to "bypass" the need for explicit (and generally complicated) cancellation between perturbative and non-perturbative contributions, at least for certain physical quantities like the mass gap. Note also that the linear DE-VIP taking the form (12), and (18) when combined with the Borel method, is only one among other possible similar resummations. In particular, we emphasize that the obtained convergence properties for $\mathrm{Re}[F] < 0$ ($\mathrm{Re}[\hat{m}_{\mathrm{v}}/\Lambda] < 0$) do not depend on the detailed properties of the "delta expansion" contour integrals here considered, e.g. (12) [which lead however to rather simple and tractable expressions in the massless limit and for Borel transforms (18)–(20)]: more generally applying the δ expansion idea in slightly different forms may replace (13) and subsequent results with an eventually different series [25], but with similar asymptotic and (Borel) convergence properties.

In summary, our construction exhibits an explicit counter-example to the conventional wisdom arguments (i) and (ii), and to some extent (iii), mentioned in the introduction. In the present paper we have only analyzed the formal Borel convergence properties, a priori applicable to any AFT, relying essentially on the properties of $F(\hat{m})$ in (3) and (7), interpolating smoothly from the ordinary perturbative coupling to the infrared mass power expansion. These convergence properties can be viewed as the generalization to $D > 1$ renormalizable theories of the ordinary convergence properties of the DE-VIP for the oscillator [9,11]. Next we argue that such a summation recipe can provide a well-defined basis to estimate more precisely some of the χSB order parameters in QCD or other models, and a more detailed study with concrete numerical applications to the GN model and QCD will be explored in [25]. Though one may eventually raise the point that in QCD-like theories, other contributions to the χSB order parameters of "truly non-perturbative" origin (i.e. unreachable by any resummation means, and/or related e.g. to instanton phenomena typically) may be expected, the resummation contributions here considered should be a useful piece of information.

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[8] The Borel summability of the exact $1/N$ $O(N)$ GN model mass gap, independently of the present construction, is analyzed in details in [27]

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